On the geometry and arithmetic of infinite translation surfaces.

Ferrán Valdez[a*] **& Gabriela Weitze-Schmithüsen**[b†]

[a] *Instituto de Matemáticas, U.N.A.M.*
Campus Morelia, Morelia, Michoacán, México
e-mail:*ferran@matmor.unam.mx*

[b] *Institute of Algebra and Geometry, Faculty of Mathematics,*
University of Karlsruhe, D-76128 Karlsruhe, Germany
e-mail:*schmithuesen@mathematik.uni-karlsruhe.de*

Abstract

We prove by constructing explicit examples that most of the classical results for number fields associated to flat surfaces fail in the realm of infinite type translation surfaces. We also investigate the relations among this fields and give a characterization for infinite type Origamis.

1 Introduction

Let S be a translation surface, in the sense of Thurston [Thu97], and denote by $\overline{S}$ the metric completion with respect to its natural translation invariant flat metric. We call a translation surface S such that $\overline{S}$ defines a covering over the flat unit torus $T = \mathbf{R}^2/\mathbf{Z}^2$ ramified over at most one point *an origami*. If S is precompact, *i.e.* when $\overline{S}$ is homeomorphic to an orientable (finite genus) closed surface, one has the following characterization:

Theorem A. [GJ00] Let S be a precompact translation surface, and let $\Gamma(S)$ be its Veech group. The following statements are equivalent.

1. The groups $\Gamma(S)$ and $\mathrm{SL}(2, \mathbf{Z})$ are commensurable.

2. Every cross-ratio of saddle connections is rational. Equivalently $K_{\mathrm{cr}}(S)$, the field of fractions of the cross-ratios of saddle connections on S, is $\mathbf{Q}$.

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3. S is (up to an affine homeomorphism) an origami.

4. There is an Euclidean parallelogram that tiles S by translations, i.e. S is a *square-tiled surface.*

The equivalence of the last two conditions is easy to sea; you can shear the parallelogram into the unit square by an affine homeomorphism. The field $K_{\mathrm{cr}}(S)$ was introduced in [GJ00] as an $\mathrm{SL}(2, \mathbf{R})$-invariant of S. Here $\mathrm{SL}(2, \mathbf{R})$ acts on the space of translation surfaces by post-composition on charts.

The first result of this article explores what remains of the preceding characterization when S is an infinite type (*i.e.* non precompact) *tame* translation surface (see §2 for a definition). More precisely:

Theorem 1. *Let S be an infinite type tame translation surface. Then,*

(A) *S is affine equivalent to an origami if and only if the set of* developed cone points *is contained in $L+x$, where $L \subset \mathbf{R}^2$ is a lattice and $x \in \mathbf{R}^2$ is fixed.*

(B) *If S is an origami the following statements (2)-(4) hold; in (1) and (5) we require in addition that there are at least two non parallel saddle connections on S:*

(1) *The Veech group of S is a subgroup of $\mathrm{SL}(2, \mathbf{Z})$.*

(2) *Suppose that S has at least four pairwise non parallel holonomy vectors. The field of cross ratios $K_{\mathrm{cr}}(S)$ is isomorphic to $\mathbf{Q}$.*

(3) *The holonomy field $K_{\mathrm{hol}}(S)$ is isomorphic to $\mathbf{Q}$.*

(4) *The saddle connection field $K_{\mathrm{sc}}(S)$ is isomorphic to $\mathbf{Q}$.*

(5) *The trace field $K_{\mathrm{tr}}(S)$ is isomorphic to $\mathbf{Q}$.*

But none of (1) - (5) implies that S is an origami.

In the proof of the preceding theorem we will show that, even if we require in (1) that the Veech group of S is equal to $\mathrm{SL}(2, \mathbf{Z})$, this condition does not imply that S is an origami. The set of *developed cone points* will be defined

in §2. For an infinite type translation surface S the Veech group $\Gamma(S)$ is formed by the differentials of orientation preserving affine automorphisms. Hence, *a priori*, $\Gamma(S)$ is a subgroup of $\mathrm{GL}_+(2, \mathbf{R})$ (*i.e.* we do not pass to $\mathrm{PGL}_+(2, \mathbf{R})$).

Furthermore, all the fields in the preceding theorem were originally defined for precompact translation surfaces. In §3 we give the definitions and extend them for infinite type tame translation surfaces. For a large class of translation surfaces, including in particular the precompact ones, we have $K_{\mathrm{tr}}(S) \subseteq K_{\mathrm{hol}}(S) \subseteq K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S)$. Observe that by theorem A the conditions (1), (2) and (4) in the second part of theorem 1 are for precompact surfaces equivalent to being an origami, whereas this is not necessarily true for (3) and (5). To examine this for (3), see the example in the proof theorem 2, §4, with $K_{\mathrm{hol}}(S) = \mathbf{Q}$ which is not an origami (and precompact); for (5) recall that a "general" translation surface has trivial Veech group, *i.e.* Veech group $\{I, -I\}$, where I is the identity matrix (see [Möl09, Thm.2.1]). In general, we just have that $K_{\mathrm{hol}}(S)$ and $K_{\mathrm{cr}}(S)$ are both subfields of $K_{\mathrm{sc}}(S)$, see §3. Thus we obtain in theorem 1 the implications $(4) \Rightarrow (2)$ and $(4) \Rightarrow (3)$. Obviously $(1) \Rightarrow (5)$ holds. We treat the remaining implications in theorem 2.

Theorem 2. *There are examples of tame translation surfaces S for which:*

  (i) *The Veech group $\Gamma(S)$ is equal to $\mathrm{SL}(2, \mathbf{Z})$ but either $K_{\mathrm{cr}}(S)$, $K_{\mathrm{hol}}(S)$ or $K_{\mathrm{sc}}(S)$ is not equal to $\mathbf{Q}$.*

  (ii) *The field $K_{\mathrm{sc}}(S)$ is equal to $\mathbf{Q}$ (hence also $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{hol}}(S)$) but $\Gamma(S)$ is not commensurable to a subgroup of $\mathrm{SL}(2, \mathbf{Z})$.*

  (iii) *$K_{\mathrm{cr}}(S)$ or $K_{\mathrm{hol}}(S)$ is equal to $\mathbf{Q}$ but $K_{\mathrm{sc}}(S)$ is not.*

  (iv) *The field $K_{\mathrm{cr}}(S)$ is equal to $\mathbf{Q}$ but $K_{\mathrm{hol}}(S)$ is not or vice versa, $K_{\mathrm{hol}}(S)$ is equal to $\mathbf{Q}$ but $K_{\mathrm{cr}}(S)$ is not.*

  (v) *The field $K_{\mathrm{tr}}(S)$ is equal to $\mathbf{Q}$ but none of the conditions (1), (2), (3) or (4) in theorem 1 hold. The conditions (3) and (4) or the condition (2) in theorem 1 hold but $K_{\mathrm{tr}}(S)$ is not isomorphic to $\mathbf{Q}$.*

The proofs of the preceding two theorems heavily rely on modifications made to construction 4.9 presented in [PSV]. One can modify such construction to prove that any subgroup of $\mathrm{SL}(2, \mathbf{Z})$ is the subgroup of an origami. From this

we can deduce the following about the oriented outer automorphism group $\mathrm{Out}^+(\mathrm{F}_2)$ of the free group F_2:

Corollary 1.1. *Every subgroup of* $\mathrm{Out}^+(\mathrm{F}_2)$ *is the stabilizer of a conjugacy class of some (possibly infinite index) subgroup of* F_2.

When S is a precompact translation surface, the existence of hyperbolic elements (*i.e.* whose trace is less than 2) in $\Gamma(S)$ has consequences for the image of $H_1(S, \mathbf{Z})$ in $\mathbf{R}^2$ under the developing map (a.k.a. *holonomy* map, see [KS00]) and for the nature of some of the fields associated to S. To be more precise, if S is precompact, the following is known:

(1) If there exists $M \in \Gamma(S)$ hyperbolic, then the holonomy field of S is equal to $\mathbf{Q}[tr(M)]$. In particular, the traces of any two hyperbolic elements in $\Gamma(S)$ generate the same field over $\mathbf{Q}$ (see [KS00], Theorem 28) .

(2) If there exists $M \in \Gamma(S)$ hyperbolic and $tr(M) \in \mathbf{Q}$, then S arises via a torus. That is, there is an elliptic curve E and a holomorphic map $f : \overline{S} \to E$ such that f is branched over torsion points on E and the holomorphic one form defining the translation structure on S is $f^*(dz)$ (see [McM03b], Theorem 9.8) . This is equivalent to S being an origami.

(3) If S is a "bouillabaisse surface" (*i.e.* if $\Gamma(S)$ contains two transverse parabolic elements), then $K_{\mathrm{tr}}(S)$ is totally real (see [HL06], Theorem 1.1). This implies that, if there exists $M \in \Gamma(S)$ hyperbolic and $\mathbf{Q}[tr(M)]$ is not totally real, then $\Gamma(S)$ does not contain any parabolic elements (see [*Ibid*], Theorem 1.2) .

(4) Let Λ and Λ_0 be the subgroups of $\mathbf{R}^2$ generated by the image under the holonomy map of $H_1(\overline{S}, \mathbf{Z})$ and $H_1(\overline{S}, \Sigma; \mathbf{Z})$, where Σ is the set of cone singularities of $\overline{S}$, respectively. If the affine automorphism group of S contains a pseudo-Anosov element, then Λ has finite index in Λ_0. (see [KS00], Theorem 30).

The third main result of this paper shows that when passing to infinite type tame translation surface there are no such consequences. For such surfaces, an element of $\Gamma(S) < \mathrm{GL}_+(2, \mathbf{R})$ will be called hyperbolic, parabolic or elliptic if its image in $\mathrm{PSL}(2, \mathbf{R})$ is hyperbolic, parabolic or elliptic respectively.

Theorem 3. *There are examples of infinite type tame translation surfaces S for which either (1), (2), (3) or (4) above do not hold.*

We remark that all tame translation surfaces S that realize the examples in the preceding theorem have the same topological type: one end and infinite genus. We refer the reader to [HS10], [HLT] or [HHW10] for recent developments concerning infinite type tame surfaces.

This paper is organized as follows. In §2 we recall the basics about tame translation surfaces (specially origamis), their singularities and possible Veech groups. In §3 we present the definitions of the fields listed in theorem 1 for infinite type tame translation surfaces. We prove that the main algebraic properties of these fields no longer hold for infinite translation surfaces. For example, we construct examples of infinite type translation surfaces for which the trace field is not a number field. Section 4 deals with the proofs of the three theorems stated in this section.

2 Preliminaries

In this section we recall some basic notions needed for the rest of the article. For a detailed exposition, we refer to [GJ00] and [Thu97].

A *translation surface S* will be a 2-dimensional real G-manifold with $G = \mathbf{R}^2 = Trans(\mathbf{R}^2)$. In particular, in this article, translation surfaces *have no cone angle singularities*. We will denote by $\overline{S}$ the metric completion of a translation surface S with respect to its natural flat metric. If $\overline{S}$ is homeomorphic to a orientable closed surface, we say that S is a *precompact translation surface*. By *translation map*, we mean a $G = \mathbf{R}^2$-map between translation surfaces. Every translation map $f : S_1 \longrightarrow S_2$ has a unique continuous extension $\overline{f} : \overline{S_1} \longrightarrow \overline{S_2}$.

Let $\pi_S : \widetilde{S} \longrightarrow S$ be the canonical projection from the universal cover of a translation surface S. From now on, $\widetilde{S}$ will be considered with the translation structure defined by π^*. For every deck transformation $\gamma \in \pi_1(S, *)$, there is a unique translation $\mathrm{hol}(\gamma)$ satisfying:

$$\mathrm{dev} \circ \gamma = \mathrm{hol}(\gamma) \circ \mathrm{dev} \tag{2.1}$$

where dev : $\widetilde{S} \longrightarrow \mathbf{R}^2$ denotes the developing map. The map hol: $\pi_1(S, *) \to$ Trans($\mathbf{R}^2$) $\cong \mathbf{R}^2$ is a homomorphism to the group Trans($\mathbf{R}^2$) of translations of $\mathbf{R}^2$. By considering the continuous extension of each map in (2.1) to the metric completion of its domain, we obtain:

$$\overline{\text{dev}} \circ \overline{\gamma} = \text{hol}(\gamma) \circ \overline{\text{dev}} \qquad (2.2)$$

The set of developed cone points of S is the subset of the plane $\mathbf{R}^2$ given by $\overline{\text{dev}}(\overline{\widetilde{S}} \setminus \widetilde{S})$. We denote it by $\widetilde{\Sigma}(S)$. We justify this nomenclature for the class of *tame* translation surfaces in the following paragraph.

Recall that precompact translation surfaces are obtained by gluing finitely many polygons along parallel edges and removing the vertices that are cone type singularities and possibly additional points. If one allows infinitely many polygons, *wild* types of singularities may occur (see e.g. [Cha04]). The surfaces there are prototypes for infinite type translations surfaces which are not "tame" . If all polygons are finite and congruent or there are at least only finitely many different congruence classes, however, at most cone type singularities (of finite or infinite angle) can occur. Translation surfaces such that $\overline{S}$ is obtained by adding to S cone type singularities (of finite or infinite angle, if any) were baptized *tame* in [PSV, Def. 2.2].
A *saddle connection* of a tame translation surface is a finite length geodesic whose extremities are cone type singularities. *Holonomy vectors* and the *Veech group* of a general tame translation surface are defined just as for precompact ones. See [*Ibid*] and references therein. Henceforth we will denote by $\Gamma(S)$ the Veech group of the tame translation surface S.

Square-tiled surfaces, i.e. surfaces obtained from glueing (possibly infinitely many) copies of the Euclidean unit square, are special examples of tame translation surfaces. Recall that they are represented by so-called *origamis*, which encode the combinatorial data that define the square-tiled surfaces. There are several equivalent definitions, see e.g. [Sch06, Section 1] where it is carried out for finite origamis. But the same definitions work as well for infinite origamis. In this article we will use the following three:

- An origami is obtained from gluing (possibly infinitely) Euclidean unit squares along their edges by translations according to the following rules: Each left edge is glued to precisely one right edge, each upper edge to precisely one lower edge and the resulting surface is connected.

- An origami is an unramified covering $p : X \to T_0$ of the once-punctured torus T_0.

- An origami is the conjugacy class of a subgroup of the free group F_2 in two generators.

The first definition precisely gives a square-tiled surface. In the second definition one obtains the translation structure on X by identifying the torus T with $\mathbf{R}^2/\mathbf{Z}^2$ and pulling it back via p. The equivalence of the second and the third definition mainly follows from the theorem of the universal covering. The free group F_2 plays the role of the fundamental group of the once punctured torus. See [Sch06, Section 1] for more details.

Allowing infinite type tame translation surfaces in general gives more liberty. In particular, it is understood which subgroups of $\mathrm{GL}_+(2, \mathbf{R})$ occur as Veech groups of this kind of surfaces. For each group occurring, explicit examples are constructed in [PSV].

Theorem B (*Ibid.*, Thm. 1.1, Thm. 1.2)**.** A countable subgroup Γ of $\mathrm{GL}_+(2, \mathbf{R})$ is the Veech group of a tame flat surface, if and only if it does not contain contracting elements, i.e. for all $M \in \Gamma$ there exists some $v \neq 0$ with $||v|| \leq ||Mv||$. The translations surface can here be chosen to be of infinite genus and with one end.

Furthermore there are only a few non countable groups that can occur which are explicitly determined in [*Ibid.*].

3 Fields associated to translation surfaces

There are four fields in the literature which are naturally associated to a translation surface S. They are called the *holonomy field* $K_{\mathrm{hol}}(S)$ (compare [KS00]), the *segment field* or *field of saddle connections* $K_{\mathrm{sc}}(S)$, the *field of cross ratios of saddle connections* $K_{\mathrm{cr}}(S)$ (compare [GJ00]) and the *trace field* $K_{\mathrm{tr}}(S)$. In the following, we extend their definitions to (possibly infinite) tame translation surfaces.

Remark 3.1. There are only three types of tame translation surfaces such that $\overline{S}$ has no cone type singularities: $\mathbf{R}^2$, $\mathbf{R}/\mathbf{Z}$ and flat torii. Tame translation surfaces with only one cone type singularity are cyclic coverings of $\mathbf{R}^2$ ramified over the origin. The cardinality of the fiber coincides with the total angle of the singularity mod 2π. For the following definition we will suppose that $\overline{S}$ has at least two cone type singularties, hence at least one saddle connection (see [PSV, Lemma 3.2]).

Definition 3.2. Let S be a tame translation surface and $\overline{S}$ the metric completion of S.

(a) (following [KS00, Section 7]) Let Λ be the image of $H_1(\overline{S}, \mathbf{Z})$ in $\mathbf{R}^2$ under the holonomy map h and let n be the dimension of the smallest $\mathbf{R}$-subspace of $\mathbf{R}^2$ containing Λ. The *holonomy field* $K_{\mathrm{hol}}(S)$ is the smallest subfield k of $\mathbf{R}$ such that $\Lambda \otimes_k k \cong k^n$.

(b) Let Σ denote the set of cone type singularities of $\overline{S}$. Using in (a), $H_1(\overline{S}, \Sigma; \mathbf{Z})$, the homology relative to the set of cone type singularities, instead of the absolute homology $H_1(\overline{S}, \mathbf{Z})$, we obtain the *segment field* or *field of saddle connections* $K_{\mathrm{sc}}(S)$.

(c) (following [GJ00, Section 5]) The *field of cross ratios of saddle connections* $K_{\mathrm{cr}}(S)$ is the field of fractions of the set of all cross ratios $(v_1, v_2; v_3, v_4)$, where the v_i's are holonomy vectors of $\overline{S}$. This field only makes sense when S has at least 4 non parallel holonomy vectors. As we will see later, for non compact tame surfaces, this is not always the case.

(d) Finally, the *trace field* $K_{\mathrm{tr}}(S)$ is the field generated by the traces of elements in the Veech group, i.e. $K_{\mathrm{tr}}(S) = \mathbf{Q}[\mathrm{tr}(A)|A \in \Gamma(S)]$.

Remark 3.3. (i) Definition (a) is equivalent to: For $n = 2$, take any two non parallel vectors $\{\vec{e_1}, \vec{e_2}\} \subset \Lambda$, then $K_{\mathrm{hol}}(S)$ is the smallest subfield k of $\mathbf{R}$ such that every element $\vec{v}$ of Λ can be written in the form $a\vec{e_1} + b\vec{e_2}$, with $a, b \in k$. For $n = 1$, any element $\vec{v}$ of Λ can be written as $a\vec{e_1}$, with $a \in K_{\mathrm{hol}}(S)$ and $\vec{e_1}$ any non zero (fixed) vector in Λ.

(ii) Recall that $\overline{S}$ may not be a topological surface, since it can have infinite angle cone singularities. But if Σ_{inf} (resp. Σ_{fin}) is the set of infinite (resp. finite) angle singularities, then $\tilde{S} = \overline{S} \backslash \Sigma_{\mathrm{inf}} = S \cup \Sigma_{\mathrm{fin}}$ is a surface, possibly of infinite genus. We furthermore have that the fundamental group $\pi_1(\overline{S})$ equals $\pi_1(\tilde{S})$. Indeed, for every infinite angle singularity $p \in \overline{S}$, there exists a small neighborhood U of p in $\overline{S}$ such that $U \setminus p$ is isometric to the "half plane" $(\{z \in \mathbf{C} | \Re(z) < \varepsilon\}, e^z dz)$, for some $\varepsilon \in \mathbf{R}$. Therefore, U and $U \setminus p$ are contractible and hence the fundamental group of $\tilde{S}$ and $\overline{S}$ coincide. It follows that $H_1(\overline{S}, \mathbf{Z}) \cong H_1(\tilde{S}, \mathbf{Z})$.

(iii) Recall that the cross ratio r of four vectors $v_1, \ldots, v_4$ with coordinates $\binom{x_i}{y_i}$ is equal to the cross ratio of the real numbers $r_1 = y_1/x_1, \ldots, r_4 = y_4/x_4$. i.e.

$$(v_1, v_2; v_3, v_4) = \frac{(r_1 - r_3) \cdot (r_2 - r_4)}{(r_2 - r_3) \cdot (r_1 - r_4)} \tag{3.3}$$

If $r_i = \infty$ for some $i = 1, \ldots, 4$, one eliminates the factors on which it appears in (3.3). For example, if $r_1 = \infty$, then $(v_1, v_2; v_3, v_4) = \frac{r_2 - r_4}{r_2 - r_3}$.

It follows directly from the definitions that $K_{\mathrm{hol}}(S) \subseteq K_{\mathrm{sc}}(S)$. Since the Veech group preserves the set of holonomy vectors, we furthermore have that, if there are at least two linearly independent holonomy vectors, then $K_{\mathrm{tr}}(S)$ is contained in $K_{\mathrm{hol}}(S)$. In the proof of theorem 2 (v) we see that the converse inclusion does not hold if all holonomy vectors are parallel. In Proposition 3.4 we see that for a large class of translation surfaces we in addition have $K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S)$. The main argument of the proof was given in [GJ00], there for precompact surfaces.

Proposition 3.4. *Let S be a (possibly infinite) tame translation surface, $\overline{S}$ its metric completion and $\Sigma \subset \overline{S}$ its set of singularities. Suppose that $\overline{S}$ has a triangulation by countably many triangles Δ_k ($k \in I$ for some index set I) such that:*

- *the set of vertices equals Σ and*

- *all Δ_k are geodesic triangles with respect to the flat metric on $\overline{S}$.*

Then we have $K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S)$.

Proof.
For the inclusion "$\subseteq$", let v_1, v_2, v_3, v_4 be the developing vectors of four saddle connections with different slopes. Since the cross ratio is invariant under the action of $\mathrm{GL}_+(2, \mathbf{R})$, we may assume that v_1 and v_2 are the standard basis vectors and their slopes are ∞ and 0. Furthermore, by remark 3.3, the slopes of v_3 and v_4 also lie in $K_{\mathrm{sc}}(S)$. Thus $(v_1, v_2; v_3, v_4)$ is in $K_{\mathrm{sc}}(S)$.
The opposite inclusion "$\supseteq$" follows from proposition 5.2 in [GJ00]. The statement there is for precompact surfaces, but the proof works in the same way, if one has that there exists a triangulation as required in proposition 3.4. More precisely, they define $V(S)$ as the $K_{\mathrm{cr}}(S)$-vector space spanned by the image of $H_1(\overline{S}, \Sigma; \mathbf{Z})$ under the holonomy map and show that it is 2-dimensional over $K_{\mathrm{cr}}(S)$. Hence $K_{\mathrm{sc}}(S) \subseteq K_{\mathrm{cr}}(S)$. $\square$

If S is a precompact translation surface of genus g, then $[K_{\mathrm{tr}}(S) : \mathbf{Q}] \leq g$. Moreover, the traces of elements in $\Gamma(S)$ are algebraic integers (see [McM03]). When dealing with tame translation surfaces, such algebraic properties do not hold in general.

Proposition 3.5. *For each $g \in \mathbf{N} \cup \{\infty\}$ there exists a tame translation surface S_g of infinite genus and one end such that $[K_{tr}(S_g) : \mathbf{Q}] = g + 1$. Moreover, $\Gamma(S)$ is generated by g hyperbolic elements whose traces are not algebraic numbers.*

Proof. Let $\{\lambda_n\}$ be an infinite sequence of **Q**-lineraly independent non algebraic numbers and define

$$G_g := \left\langle \left(\begin{array}{cc} \mu_n & 0 \\ 0 & \mu_n^{-1} \end{array} \right) \mid \mu_n + \mu_n^{-1} = \lambda_n \right\rangle_{n=1}^g$$

The surface S_g is the result of the construction 4.9 in [PSV] applied to G_g.

□

By letting $g = 1$ and $\lambda_1 = \pi$ we obtain the following:

Corollary 3.6. *There are examples of tame translation surfaces S of infinite genus and one end with cyclic hyperbolic Veech group and such that $K_{tr}(S)$ is not a number field.*

Trascendental numbers naturally appear also in fields associated to Veech groups arising from a generic triangular billiard. Indeed, let $\mathcal{T} \subset \mathbf{R}^2$ denote the space of triangles parametrized by two angles (θ_1, θ_2). Remark that $\mathcal{T}$ is a symplex. For every $T \in \mathcal{T}$, a classical construction due to Katok and Zemljakov produces a tame flat surface S_T from T [ZK75]. If T has an interior angle which is not commesurable with π, $\overline{S_T} \setminus \Sigma_{\inf}$ has infinite genus and one end. This kind of topological surface is also known as *the Loch Ness Monster* [Val09].

Proposition 3.7. *The set $\mathcal{T}' \subset \mathcal{T}$ formed by those triangles such that $K_{sc}(S_T)$, $K_{cr}(S_T)$ and $K_{tr}(S_T)$ are not number fields is of total (Lebesgue) measure in $\mathcal{T}$.*

Proof. Since S_T has a triangulation with countably many triangles satisfying the hypotheses of proposition 3.4, the fields $K_{sc}(S_T)$ and $K_{cr}(S_T)$ coincide. Without loss of generality we can suppose that $\{1, \rho e^{i\theta_1}, e^{2i\theta_1}\}$, for some $\rho > 0$, are holonomy vectors in Λ. Choose the base of $\mathbf{R}^2$ given by $\{1, \rho e^{i\theta_1}\}$. Then, solutions for $a + b\rho e^{i\theta_1} = e^{2i\theta_1}$ are given by:

$$a = -1 \qquad b = \frac{2\cos\theta_1}{\rho}$$

Therefore $\frac{2\cos\theta_1}{\rho}$ is an element of $K_{sc}(S)$. Furthermore, from [Val09b], we know that the rotation matrix:

$$\left(\begin{array}{cc} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{array} \right)$$

is in $\Gamma(S_T)$. Hence $2\cos\theta_1 \in K_{tr}(S_T)$.

□

4 Proof of main results

In this section we prove the results stated in the introduction.

Proof Theorem 1. We begin proving part (A). Denote by $T_0 = T \setminus \{\infty\}$ the once-punctured torus $T = \mathbf{R}^2/L$ so that $\rho : S \longrightarrow T_0$ is an origami. Then, the following diagram commutes, for ρ and π_S are translation maps:

$$\begin{array}{ccc}
\widetilde{S} & \hookrightarrow & \overline{\widetilde{S}} \\
\downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \overline{\pi_S}} \quad \searrow^{\overline{dev}} \\
S & \hookrightarrow & \overline{S} \qquad \mathbf{R}^2 \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \overline{\rho}} \quad \swarrow_{\pi_T} \\
T_0 & \hookrightarrow & T
\end{array} \qquad (4.4)$$

Given that $T \setminus T_0 = \infty = \overline{\rho} \circ \overline{\pi_S}(\overline{\widetilde{S}} \setminus \widetilde{S})$, the projection of $\widetilde{\Sigma}(S)$ to T is just a point. This proves sufficiency.

Equation (2.2) implies that, if $\widetilde{\Sigma}(S)$ is contained in $L + x$, then every $\mathrm{hol}(\gamma)$ is a translation of the plane of the form $z \to z + \lambda_\gamma$, where $\lambda_\gamma \in L$. Puncture $\widetilde{S}$ and S at $\mathrm{dev}^{-1}(\mathrm{L}+\mathrm{x})$ and $\pi_S(\mathrm{dev}^{-1}(\mathrm{L}+\mathrm{x}))$ respectively to obtain $\widetilde{S}_0$ and S_0. Denote $\mathbf{R}^2_0 = \mathbf{R}^2 \setminus L + x$. Given that $\widetilde{S}_0$ has the translation structure induced by pullback of (the restriction of) π_S, the map $\mathrm{dev}_| : \widetilde{S}_0 \longrightarrow \mathbf{R}^2_0$ is a flat surjective map (see [Thu97], §3.4). Equation (2.1) implies that:

$$\begin{array}{ccc}
\widetilde{S}_0 & \xrightarrow{\mathrm{dev}_|} & \mathbf{R}^2_0 \\
\downarrow{\scriptstyle \pi_{S|}} & & \downarrow{\scriptstyle \pi_{T|}} \\
S_0 & & T_0
\end{array} \qquad (4.5)$$

descends to a flat covering map $\rho : S_0 \longrightarrow T_0$. Hence $\overline{S} = \overline{S_0}$ defines a covering over a flat torus ramified over at most one point. This proves necessity.

(B) Let $p : \overline{S} \to T$ be an origami, i.e. ramified at most over $\infty \in T$. All saddle connections of S are preimages of closed simple curves on T with base point at ∞. This implies that the derivatives of all saddle connections are vectors with integer coordinates. Thus $K_{\mathrm{sc}}(S) = K_{\mathrm{hol}}(S) = \mathbf{Q}$. If S has at least four non parallel holonomy vectors, then furthermore $K_{\mathrm{cr}}(S) = \mathbf{Q}$. If S has at least two linearly independent holonomy vectors, then the Veech group must preserve the lattice spanned by them. Thus it is commensurable

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to a (possible infinite index) subgroup of $\mathrm{SL}(2, \mathbf{Z})$.

Hence every origami fulfills conditions (2), (3) and (4). If it has at least two linearly independent holonomy vectors, then it fulfills in addition (1) and (5). Contrariwise, the proof of Theorem 2 (i) gives an example for a translation surface S with Veech group $\mathrm{SL}(2, \mathbf{Z})$ that has four non parallel holonomy vectors. The segment field in this case $K_{\mathrm{sc}}(S)$ is not $\mathbf{Q}$, thus it is not and origami. Hence it is an example for the fact that (1) and (5) do not imply that the translation surface is an origami. In Theorem 2 (ii) the Veech group of the translation surface S constructed in the proof is $\mathrm{SL}(2, \mathbf{Q})$ and there are two non parallel holonomy vectors. Thus S contradicts (1), which implies that S is not an origami. But $K_{\mathrm{cr}}(S) = K_{\mathrm{hol}}(S) = K_{\mathrm{sc}}(S) = K_{\mathrm{tr}}(S) = \mathbf{Q}$. Hence (2) - (5) also do not imply that the translation surface is an origami.

Proof Theorem 2. The proof of this theorem uses construction 4.9 presented in [PSV]. We strongly recommend readers unfamiliar with this construction to read it first, in order to grasp the key ideas of the proof.

We first show how to construct a tame translation surface such that (i) holds. It is sufficient to show that (i.i) there exists S such that $\Gamma(S) = \mathrm{SL}(2, \mathbf{Z})$ but $[K_{\mathrm{cr}}(S) : \mathbf{Q}] > 1$ or (i.ii) such that $\Gamma(S) = \mathrm{SL}(2, \mathbf{Z})$ but $[K_{\mathrm{hol}}(S) : \mathbf{Q}] > 1$.

We begin by proving (i.i). Let $G = \mathrm{SL}(2, \mathbf{Z})$. Apply construction 4.9 in [PSV] to G but choose the family of marks $\mathcal{C}^{-1}$ employed in this construction in such a way that the there exists $N \in \mathbf{Z}$ and $\alpha > 0$ irrational so that (α, N) is a holonomy vector. This is possible for in the cited construction the choice of the point (x_1, y_1) is free. This way, we obtain a surface S whose Veech group $\Gamma(S)$ is $\mathrm{SL}(2, \mathbf{Z})$. Observe that $v_1 = (-1, 1)$, $v_2 = (0, 1)$, $v_3 = (1, 0)$ and $v_4 = (\alpha, N)$ are holonomy vectors of $\overline{S}$. Let l_i be lines in $\mathbf{P}^1(\mathbf{R})$ containing v_i, $i = 1, \ldots, 4$ respectively. A direct calculation shows that the cross ratio of these four lines is $\frac{\alpha}{\alpha+N}$, which lies in $K_{\mathrm{cr}}(S)$.

The example showing (i.ii) also results from a slight modification made to construction 4.9 in [PSV]. Let $\{\vec{e}, \vec{f}\}$ be the standard basis of $\mathbf{R}^2$. There exists a natural number $n > 0$ such that the mark M in A_{Id} whose endpoints are $-n\vec{e}$ and $-(n-1)\vec{e}$ does not intersect all other marks used in the construction. On a $\pi \times e$ rectangle R, where e is Euler's number, identify opposite sides to obtain a flat torus T. Consider on T a horizontal mark M' of length 1 and glue A_{Id} with T along M and M'. Then proceed with construction 4.9 in a $\mathrm{SL}(2, \mathbf{Z})$-equivariant way as indicated in [*Ibid.*] to obtain a tame flat surface S whose Veech group is $\mathrm{SL}(2, \mathbf{Z})$. The image of $H_1(\overline{S}, \mathbf{Z})$ under the holonomy map contains the vectors $\vec{e}$, $e \cdot \vec{e}$ and $\pi \cdot \vec{f}$. Hence $[K_{\mathrm{hol}}(S) : \mathbf{Q}] > 1$.

To obtain S such that (ii) holds consider $G = \mathrm{SL}(2, \mathbf{Q})$ or $G = \mathrm{SO}(2, \mathbf{Q})$. These are non discrete countable subgroups of $\mathrm{SL}(2, \mathbf{R})$, hence without con-

tracting elements and not commensurable to a subgroup of $\mathrm{SL}(2, \mathbf{Z})$. Then apply construction 4.9 in [PSV] to any of these groups but choose the points (x_i, y_i) defining the families of marks $\mathcal{C}^i$ in $\mathbf{Q} \times \mathbf{Q}$. This way we obtain a surface S whose Veech group is isomorphic to G and whose holonomy vectors S have all coordinates in $\mathbf{Q} \times \mathbf{Q}$. This implies that $K_{\mathrm{sc}}(S)$ is $\mathbf{Q}$.

We now prove (iii). First we construct S such that $K_{\mathrm{cr}}(S) = \mathbf{Q}$ but $K_{\mathrm{sc}}(S)$ is not, as follows. Let P_1, P_2 and P_3 be three copies of $\mathbf{R}^2$, choose on each copy an origin and let $\{\vec{e}, \vec{f}\}$ be the standard basis. Consider:

1. Marks v_n on the plane P_1 along segments whose endpoints are $n \cdot \vec{f}$ and $n \cdot \vec{f} + \vec{e}$ with $n = 0, 1$.

2. Marks on P_2 and P_3 along the segments w_0, w_1 whose endpoints are $(0,0)$ and $(1,0)$. Then along the segments z_0 and z_1 whose endpoints are $(2,0)$ and $(2 + \sqrt{p}, 0)$, for some prime p.

Glue the three planes along slits as follows: v_i to w_i, for $i = 0, 1$ and z_0 to z_1. The result is the surface S satisfying (iii) and having six conic singularities, each of total angle 4π. Indeed, $\{0, 1, -1, \infty\}$ parameterizes all possible slopes of lines through the origin in $\mathbf{R}^2$ containing holonomy vectors. Hence $K_{\mathrm{cr}}(S) = \mathbf{Q}$. On the other hand, the set of holonomy vectors contains $(1,0)$, $(0,1)$, $(1,1)$ and $(0, \sqrt{p})$. Therefore $K_{\mathrm{sc}}(S)$ contains $\mathbf{Q}(\sqrt{p})$ as subfield.

We finish the proof of (iii) by constructing a precompact tame translation surface such that $K_{\mathrm{hol}}(S) = \mathbf{Q}$ but $K_{\mathrm{sc}}(S)$ is not. Consider two copies L_1 and L_2 of the L-shaped Origami tiled by 3 unit squares (see e.g. [HL06b, Example on p.293]). Consider a point $p_i \in L_i$ at distance $0 < \varepsilon << 1$ from the 6π-angle singularity s_i, $i = 1, 2$. Let m_i be a marking of length ε on L_i defined by a geodesic of length ε joining p_i to s_i, $i = 1, 2$. We can choose p_i so that the angle between $h(m_i)$ and h(any side of a square) is irrational multiple of π and both marking are parallel. Then glue L_1 and L_2 along m_1 and m_2 to obtain S. By construction $h(H_1(\overline{S}, \mathbf{Z})) = \mathbf{Z} \times \mathbf{Z}$ but $h(H_1(\overline{S}, \Sigma; \mathbf{Z}))$ contains an orthonormal base $\{\vec{e}, \vec{f}\}$ and a vector $h(m_1)$ such that neither $\angle \vec{e} h(m_1)$ nor $\angle \vec{f} h(m_1)$ are rational. This implies that $K_{\mathrm{sc}}(S)$ is not isomorphic to $\mathbf{Q}$.

We address (iv) now. The surface S constructed for (iii) such that $K_{\mathrm{hol}}(S) = \mathbf{Q}$ but $K_{\mathrm{sc}}(S)$ is not also satisfies that $K_{\mathrm{cr}}(S)$ is not equal to $\mathbf{Q}$. Indeed, it is sufficient to remark again that neither $\angle \vec{e} h(m_1)$ nor $\angle \vec{f} h(m_1)$ are rational. We now construct S such that $K_{\mathrm{cr}}(S) = \mathbf{Q}$ but $K_{\mathrm{hol}}(S)$ is not. Take two copies of the real plane P_1 and P_2. Choose an origin and let $\vec{e}, \vec{f}$ be the standard basis. Let $\mu_i > 1$, $i = 1, 2, 3$ be three distinct irrational numbers and define $\lambda_0 = 0$ and $\lambda_n = \sum_{i=1}^{n} \mu_i$ for $n = 1, \ldots, 3$. On P_1 consider the markings m_n whose endpoints are $n\vec{f}$ and $n\vec{f} + \vec{e}$ for $n = 0, \ldots, 3$. On P_2

consider the markings m'_n whose endpoints are $(n + \lambda_n)\vec{e}$ and $(n + \lambda_n + 1)\vec{e}$ for $n = 0, \ldots, 3$. Glue P_1 and P_2 along the markings m_n and m'_n. The result is a tame flat surface S with eight 4π-angle singularities. These singularities lie on P_2 on a horizontal line and hence we can naturally order them from, say, left to right. Let's denote these ordered singularities by a_j, $j = 1, \ldots, 8$. Let $g_{\vec{e}}(a_i, a_j)$ (respectively $g_{\vec{f}}(a_i, a_j)$) be the directed geodesic in S parallel to $\vec{e}$ (respectively $\vec{f}$) joining a_i with a_j. Define in $H_1(\overline{S}, \mathbf{Z})$:

- The cycle c_1 as $g_{\vec{e}}(a_3, a_4)g_{\vec{e}}(a_4, a_5)g_{\vec{f}}(a_5, a_3)$.

- The cycle c_2 as $g_{\vec{e}}(a_4, a_3)g_{\vec{e}}(a_3, a_2)g_{\vec{f}}(a_2, a_4)$.

- The cycle c_3 as $g_{\vec{f}}(a_6, a_8)g_{\vec{e}}(a_8, a_7)g_{\vec{e}}(a_7, a_6)$.

where the product is defined to be the composition of geodesics on S. Remark that $h(c_1) = (1 + \mu_2, -1)$, $h(c_2) = (-(1 + \mu_1), 1)$ and $h(c_3) = (-(1 + \mu_3), 1)$. We can choose parameters μ_i, $i = 1, 2, 3$ so that $\mathbf{Z}$-module generated by these 3 vectors has rank 3. Therefore $K_{\mathrm{hol}}(S)$ cannot be isomorphic to $\mathbf{Q}$.

We address now (v). We construct first a flat surface S for which $K_{\mathrm{tr}}(S) = \mathbf{Q}$ but none of the conditions (1), (2), (3) or (4) in theorem 1 hold. In the proof of theorem 1, part (i.ii), we constructed a surface S such that $\Gamma(S) = \mathrm{SL}(2, \mathbf{Z})$ but $[K_{\mathrm{hol}}(S) : \mathbf{Q}] > 1$. We modify this surface to obtained the desired surface in the following way. First, change $\mathrm{SL}(2, \mathbf{Z})$ for $\mathrm{SL}(2, \mathbf{Q})$. Second, let the added mark M be of unit length and such that the vector defined by developing it along the flat structure does not lie in the lattice $\pi\mathbf{Z} \times e\mathbf{Z}$ nor has rational slope. The result of this modification is a tame translation surface S homeomorphic to the Loch Ness Monster for which $\Gamma(S) = \mathrm{SL}(2, \mathbf{Q})$ and for which both $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{hol}}(S)$ (hence $K_{\mathrm{sc}}(S)$ as well) have degree at least 1 over $\mathbf{Q}$.

We now describe the surface S satisfying conditions (3) and (4) in theorem 1 but such that $K_{\mathrm{tr}}(S)$ is not isomorphic to $\mathbf{Q}$. In [PSV] there is an explicit examples of a tame translation surface S homeomorphic to the Loch Ness monster such that $\Gamma(S)$ is the group of matrices:

$$\begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} \text{ , where } t \in \mathbf{R}, \ s \in \mathbf{R}^+$$

The surface S is obtained by glueing two copies of the plane along two infinite families of marks, each family contained in a line and formed by marks of length 1. Hence, w.l.o.g. , we can suppose that the image under the holonomy map of $H_1(\overline{S}, \mathbf{Z})$ and $H_1(\overline{S}, \Sigma; \mathbf{Z})$ both lie in $\mathbf{Z}$. Therefore $K_{\mathrm{hol}}(S) = K_{\mathrm{sc}}(S) = \mathbf{Q}$, but $K_{\mathrm{tr}}(S) = \mathbf{R}^+$. Remark that all holonomy vectors of $\overline{S}$ in this example are parallel.

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Finally, we construct a tame translation surface S such that $K_{\mathrm{cr}}(S) = \mathbf{Q}$ but $K_{\mathrm{tr}}(S)$ is not equal to $\mathbf{Q}$. Let E_0 be a copy of the affine plane $\mathbf{R}^2$ with a chosen origin and (x, y)-coordinates. Slit E_0 along the rays $R_v := (0, y \geq 1)$ and $R_h := (x \geq 1, 0)$ to obtain $\hat{E}_0$. Choose $0 < \lambda < 1$ irrational and $n \in \mathbf{N}$ so that $1 < n\lambda$. Define:

$$
M := \begin{pmatrix} \lambda & 0 \\ 0 & n\lambda \end{pmatrix} \qquad R_v^k := M^k R_v \qquad R_h^k := M^k R_h \quad k \in \mathbf{Z}
$$

Here M^k acts linearly on E_0. For $k \neq 0$, slit a copy of E_0 along the rays R_v^k and R_h^k to obtain $\hat{E}_k$. We glue the family of slitted planes $\{\hat{E}_k\}_{k \in \mathbf{Z}}$ to obtained the desired tame flat surface as follows. Each $\hat{E}_k$ has a "vertical boundary" formed by two vertical rays issuing from the point of coordinates $(0, (n\lambda)^k)$. Denote by $R_{v,l}^k$ and $R_{v,r}^k$ the boundary ray to the left and right respectively. Identify by a translation the rays $R_{v,r}^k$ with $R_{v,l}^{k+1}$, for each $k \in \mathbf{Z}$. Denote by $R_{h,b}^k$ and $R_{h,t}^k$ the horizontal boundary rays in $\hat{E}_k$ to the bottom and top respectively. Identify by a translation $R_{h,b}^k$ with $R_{h,t}^{k+1}$ for each $k \in \mathbf{Z}$. By construction, $\{(-\lambda^k, (n\lambda)^k)\}_{k \in \mathbf{Z}}$ is the set of all holonomy vectors of S. Clearly, all slopes involved are rational, hence $K_{\mathrm{cr}}(S) = \mathbf{Q}$. On the other hand, $M \in \Gamma(S)$ and $tr(M) = (n+1)\lambda$. Remark that the surface S constructed in this last paragraph admits not triangulation satisfying the hypotheses of proposition 3.4.

$\square$

Proof of Corollary 1.1:
Let Γ be a subgroup of $\mathrm{SL}_2(\mathbf{Z})$. By Theorem B §2 we know that there is a translation surface S with Veech group Γ. Furthermore in the construction 4.9 in [PSV] all slits can be chosen such that their end points are integer points in the corresponding plane $\mathbf{C} = \mathbf{R}^2$, thus S is an origami, i.e. it allows a covering $p : \overline{S} \to T$ ramified at most over one point ∞. Recall that p defines the conjugacy class $[U]$ of a subgroup U of F_2 as follows. The group U is the fundamental group of $S^* = \overline{S} \backslash p^{-1}(\infty)$. It is embedded into $F_2 = \pi_1(E^*)$ via the homomorphism p^* between fundamental groups which is induced by p. The embedding depends on the choices of the base points up to conjugation. In [Sch04] this is used to give the description of the Veech group cited in Theorem C completely in terms of $[U]$. Recall for this that the outer automorphism group $\mathrm{Out}(F_2)$ is isomorphic to $\mathrm{GL}_2(\mathbf{Z})$. Furthermore it naturally acts on the set of the conjugacy classes of subgroups U of F_2.

Theorem C ([Sch04], Prop. 2.1)**.** The Veech group $\Gamma(S^*)$ equals the stabilizer of the conjugacy class $[U]$ in $\mathrm{SL}_2(\mathbf{Z})$ under the action described above.

The theorem in [Sch04] considers only finite origamis, but the proof works in the same way for infinite origamis. Recall furthermore that $\Gamma(S^*) = \Gamma(S) \cap \mathrm{SL}_2(\mathbf{Z})$ and $\Gamma(S) \subseteq \mathrm{SL}_2(\mathbf{Z})$ if and only if the $\mathbf{Z}$-module spanned by the holonomy of the saddle connections equals $\mathbf{Z}^2$. But we can easily choose the slits in the construction in [PSV] such that this condition is fulfilled.

$\square$

Proof of Theorem 3:

First remark that the translation surfaces constructed in the proof of theorem 2 parts (i.ii) and (v) are counterexamples for statements (1) and (2) repectively. Moreover, it is easy to construct using construction 4.9 in [PSV] an example of a translation surface S whose Veech group is generated by two hyperbolic elements M_1 and M_2 but such that $\mathbf{Q}(tr(M_1))$ and $\mathbf{Q}(tr(M_2))$ are not the same field (compare statement (1) in theorem 3). On the other hand, if μ is a solution to the equation $\mu + \mu^{-1} = \sqrt[3]{2}$ and G is the group generated by the matrices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}. \tag{4.6}$$

then construction 4.9 in [PSV] applied to G produces a tame translation surface that is a counterexample for (3).

Finally we construct a tame translation surface S with a hyperbolic element on its Veech group and for which Λ has no finite index in Λ_0. The construction has two steps.

Step 1: Let M be the matrix given by

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{4.7}$$

Let S' be the tame translation surface obtained from performing construction 4.9 in [PSV] to the group G' generated by M. Let Λ' be the image in $\mathbf{R}^2$ under the holonomy map of $H_1(\overline{S'}, \mathbf{Z})$, $\{\vec{e}, \vec{f}\}$ be the standard basis of $\mathbf{R}^2$ and $\beta := G'\{\vec{e}, \vec{f}\}$ (each matrix acting linearly on $\mathbf{R}^2$). We suppose w.l.o.g. that $\vec{e}$ lies in Λ'. Let $V \subset \mathbf{R}^2$ be the subgroup of $\mathbf{R}^2$ generated by $\Lambda' \cup \beta$.

Step 2: Let $\alpha = \{v_j\}_{j \in \mathbf{N}} \subset \mathbf{R}^2 \setminus V$ be a sequence of $\mathbf{Q}$-linearly independent vectors. We modify construction 4.9 in [PSV] applied to G' using α to obtain S in the following way. We add to the page A_{Id} a family of marks parallel to vectors in α. We can suppose that this family of marks lie in the left half plane $Re(z) < 0$ in A_{Id}, are disjoint by pairs and do not intersect any of the marks in $\mathcal{C}_1$ used in construction 4.9 applied to G'. For each $j \in \mathbf{N}$ there exists a natural number k_j such that $2k_j > |v_j|$. Let T_j be the torus obtained from a $2k_j \times 2k_j$ square by identifying opposite sides. Slit each T_j along a

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vector parallel to v_j and glue it to A_{Id} along the mark parallel to v_j. Denote by A'_{Id} the result of performing this operation for every $j \in \mathbf{N}$. Then proceed with construction 4.9 applied to G' as indicated in [PSV] to obtain S.

Let Λ be the image in $\mathbf{R}^2$ under the holonomy map of $H_1(S, \mathbf{Z})$, Λ_1 the subgroup of $\mathbf{R}^2$ generated by $\alpha \cup V$ and Λ_0 the image under the holonomy map of relative homology. Then, by construction, $\Lambda \lhd \Lambda_1 \lhd \Lambda_0$ and the index of Λ in Λ_0 cannot be smaller than the cardinality of α, which is infinite. $\quad\square$

References

[Cha04]   R. Chamanara, *Affine automorphism groups of surfaces of infinite type*, Contemporary Mathematics **355** (2004), 123–145.

[GJ00]   E. Gutkin and C. Judge, *Affine mappings of translation surfaces: geometry and arithmetic*, Duke Math. J. **103** (2000), no. 2, 191–213.

[HHW10]   P. Hooper, P. Hubert, and B. Weiss, *Dynamics on the infinite staircase surface* (2010), available at http://www.math.bgu.ac.il/~barakw/papers/staircase.pdf.

[HL06a]   P. Hubert and E. Lanneau, *Veech groups without parabolic elements*, Duke Math. J. **133** (2006), no. 2, 335–346.

[HL06b]   P. Hubert and S. Lelièvre, *Prime arithmetic Teichmüller discs in $\mathcal{H}(2)$*, Isr. J. Math. **151** (2006), 281–321.

[HLT]   P. Hubert, S. Lelièvre, and S. Troubetzkoy, *The Ehrenfest wind-tree model: periodic directions, recurrence, diffusion*, to appear in J. Reine Angew. Math., available at http://www.cmi.univ-mrs.fr/~hubert/articles/windtree.pdf.

[HS10]   P. Hubert and G. Schmithüsen, *Infinite translation surfaces with infinitely generated Veech groups*, J. Mod. Dyn. **4** (2010), no. 4, 715–732.

[KS00]   R. Kenyon and J. Smillie, *Billiards on rational-angled triangles*, Commentarii Mathematici Helvetici **75** (2000), no. 1, 65–108.

[ZK75]   A. Zemljakov and A. Katok, *Topological transitivity of billiards in polygons*, Mat. Zametki **18** (1975), no. 2, 291–300.

[McM03a]   C. McMullen, *Billiards and Teichmüller curves on Hilbert modular surfaces*, J. Amer. Math. Soc. **16** (2003), no. 4.

[McM03b]   ———, *Teichmüller geodesics of infinite complexity*, Acta Math. **191** (2003), no. 2, 191–223.

[Möl09]   M. Möller, *Affine groups of flat surfaces*, Papadopoulos, Athanase (ed.), Handbook of Teichmüller theory Volume II, Chapter 10 (EMS), IRMA Lectures in Mathematics and Theoretical Physics 13 (2009), 369–387.

[PSV]   P. Przytycki, G. Schmithüsen, and F. Valdez, *Veech groups of Loch Ness Monsters*, Ann. Inst. Fourier (Grenoble), *to appear*.

[Sch04]   G. Schmithüsen, *An algorithm for finding the Veech group of an origami*, Experimental Mathematics **13** (2004), no. 4, 459–472.

[Sch06]   _____, *Origamis with non congruence Veech groups.*, Proceedings of Symposium on Transformation Groups, Yokohama (2006).

[Thu97]   W. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[Val09a]   F. Valdez, *Infinite genus surfaces and irrational polygonal billiards*, Geom. Dedicata **143** (2009), 143–154.

[Val09b]   _____, *Veech groups, irrational billiards and stable abelian differentials.* (2009), available at `http://arxiv.org/abs/0905.1591v2`.